Rhetorical Analysis of Mathematical Organization and Style in “Three dimensions of knot coloring”

by Max Ristenberg

Introduction

Mathematics is similar to other sciences in its demand for precision, rigor, and strict logic. However, it takes these characteristics to the extreme, entering an abstract world, which results in slightly different demands on its methods of communication. Instead of the compartmentalized IMRaD structure, mathematical papers tend to staircase, building up the conceptual theory section by section, and including the results alongside the methods used to discover them. This strategy unifies what mathematicians discover with how they think about problems; oftentimes, the perspective is even more important than the results.

“Three dimensions of knot coloring” by Carter, Silver, and Williams is an article on the problem of knot coloring published in the American Mathematical Monthly in 2014. A knot is a closed loop in space; the way it twists and crosses over itself encodes its relationship to the ambient space. Knots are closely related to many deep problems in low-dimensional topology and geometry, but their concrete, almost playful nature makes them accessible to advanced undergraduates. It is likely that Carter, Silver, and Williams anticipate an audience of aspiring knot theorists, and aim to inspire their readers to study these advanced topics. To that end, they have chosen an especially linear structure to their paper.

In addition to the organization, the style of “knot coloring” is carefully crafted so that the paper will be as readable as possible. This is evidences by the authors’ choices in the level of rigor in the primary arguments of the paper. In mathematics, rigor describes the level of detail used for proofs. A highly rigorous proof explicitly describes each step of the proof, while a brief sketch of the main ideas behind an argument characterizes a low level of rigor. “Knot coloring” uses a high level of rigor for a majority of the paper, which means that a reader will not have to think as hard to fill in missing details. Unlike a typical research-level publication, the direct style in “knot coloring” makes it accessible to a wider, more novice audience.

Background and Summary

In “knot coloring,” knots are represented by diagrams lying in the plane. A coloring in mathematics is just a way of associating distinct “colors” to an object, so it suffices to associate numbers to the knot. (One can always decide to take “0” to mean “red,” “1” to mean “blue,” etc., but since Carter, Silver and Williams want to generalize to very many colors, it’s easiest to think of these as numbers.) Actually, in this context, colorings are more special; they indicate ways that the knot can be rotated and flipped through space. To ensure this extra structure, particular numerical restraints are placed on the colorings.

The paper has four main sections: Introduction, Fox and Dehn colorings, Alexander-Briggs colorings, and Taking knot colorings in other directions. The Introduction of “knot coloring” establishes the history of the mathematical study of knots, from Peter Guthrie Tait’s early formulation of knots to Kurt Reidemeister's major theorem in 1926 regarding what are now called Reidemeister moves, and beyond to the Jones polynomial, a knot invariant introduced by V.F.R. Jones and interpreted by L.H. Kauffman in the mid-1980s. The introduction concludes with a structured overview of the organization of the paper.

When knots are represented by planar diagrams, arcs pass over each other at locations called crossings. When knots are represented via the tromp l'oeil device that goes back to Tait, it looks like one arc is hidden by another at a crossing. The strands of a knot diagram are the parts that can be drawn without lifting a pencil; they are the arc segments between undercrossings. Fox colorings are colorings of the strands obeying the Fox n-coloring rule. The first half of the second section establishes some theory of the Fox colorings, including the relationship between Fox colorings and certain symmetries of the knot resembling rotations and flips.

The authors continue by defining Dehn n-colorings which are colorings of the regions of the knot diagram, rather than the strands. Again, Dehn colorings are required to obey a numerical rule, which guarantees that the colorings carry information of symmetries of the knot. The key observation is that while the colorings seem to be of entirely distinct natures, they carry precisely the same information: the symmetries of rotations and flips. Using an analogy from multivariable calculus, the authors close the second connection by explicitly connecting Dehn n-colorings and Fox n-colorings.

Planar diagram of a knot Fox 5-coloring of a knot

In the third section, Carter, Silver and Williams describe Alexander-Briggs colorings. These are vertex colorings, which again are required to satisfy a special restraint which guarantees a connection with Dehn colorings. This restraint is more easily expressed using Tait diagrams, which carry the same information as other knot diagrams, but rather than "hiding" the lower arc, dots indicate which arc is passing over the other. Alexander-Briggs colorings are more obscure than the other colorings, and demonstrating their equivalence with the other colorings constitutes the primary result of the paper.

The final section quickly describes how colorings carry information related to various sophisticated or obscure mathematical objects, without details explaining precisely what the new objects are.

The Staircase model

Steven Kleiman, a Professor of Mathematics at MIT, advises mathematical authors to "present the material in small digestible portions... if possible, follow a sequential path through the subject." Carter, Silver and Williams follow this model closely, and the result is an overall very understandable paper.

The main point of the paper is that, after ignoring certain redundancies, Fox, Dehn, and Alexander-Briggs colorings are all equivalent. That means if you give a knot a Fox coloring, there is a Dehn coloring and an Alexander-Briggs coloring implicitly attached. That is, if you color the strands of the knot, somehow you have implicitly colored the regions and the vertices as well.

In order to express the three-way correspondence, the authors have to define each of the three colorings, and then express at least two correspondences between them. They make the choice to begin by defining the Fox colorings, the most well-known of the colorings. They go on to define Dehn colorings, since they have an explicit correspondence from Fox to Dehn colorings. This constitutes a section in its entirety, since the correspondence is a mathematical result, stated as a theorem.

Dehn 5-coloring Alexander-Briggs 5-coloring

In the third section, Alexander-Briggs colorings are introduced, and the correspondence between Dehn colorings and Alexander-Briggs colorings is established. This completes the primary goals of the paper, because the correspondence between Fox colorings and Alexander-Briggs colorings is a logical consequence of the correspondences now explicitly proven.
The authors had good reason to begin with the Fox and Dehn colorings. These two forms of knot coloring are certainly the most famous of the three. They are also the easiest to define, since their coloring rules may be easily expressed using standard knot diagrams with the "hidden line" device. Alexander-Briggs colorings are more difficult to deal with, because they demand use of the Tait diagram, which needs to be slightly modified for their second theorem. The amount of prose necessary to establish the results regarding Alexander-Briggs colorings must have compelled the authors to devote a section to them.

The final section comes after the main results of the paper have been adequately established and discussed. This frees the authors to briefly describe some new objects such as such as branched covers, quandles, "infinite colorings." The terse descriptions contrast with the details in previous sections, but this is not necessarily a flaw in the paper. By complementing understandable results with more sophisticated ideas, the organization of "knot coloring" acts as a launch pad for young eager knot theorists.

Level of Rigor

While in most of the sciences, validity and importance of results follows from statistical calculations, mathematics claims truth through its strict adherence to deductive logic. However, as mathematical objects become more complex, it often becomes burdensome to spell out every detail. As a result, mathematicians have to consider what level of rigor to employ when writing; this is main dynamic behind choices of directness and concision in mathematical writing.

Consider the following passages from "knot coloring:"

> "Given a Dehn n-coloring, it is easy to obtain a Fox n-coloring: assign to each arc the sum of the colors of two bounding regions separated by that arc." (page 6)
>
> "An elementary argument using Reidemeister moves shows that the number of

> Fox n-colorings does not depend on the specific diagram for [the knot] that we choose. ... With a bit more work, one sees that the set of Fox n-colorings forms a module over the ring Z/nZ." (page 4)

The disparity in detail is clear. The first passage gives a very explicit way to see the relationship between Dehn colorings and Fox colorings. Just take the literal numbers on the diagram, follow this recipe, and write down some more numbers on the diagram. The latter passage is far more conceptual. It does not give a recipe, or even a strategy for proof. It merely states facts with the slightest comments about how to properly justify them.

It might seem preferable to demand a high standard of rigor for all mathematics, but the latter passage from "knot coloring" is typical of mathematical research articles. Terence Tao, a Professor of Mathematics at UCLA well-known for his blog on mathematical writing, says "if you have just learnt how to prove a standard lemma which is well known to the experts and already in the literature, this does not mean that you should provide the standard proof of this standard lemma, unless this serves some greater purpose in the paper." Authors should use their paper to provide details for their innovations, rather than regurgitating results familiar to experts in the field. Professor Tao suggests that "A paper should dwell at length (using plenty of English) on the most important, innovative, and crucial components of the paper."

In fact, Carter, Silver and Williams are generally extremely clear, concrete and direct. The bulk of the paper more closely resembles the first passage than the second. A typical mathematical research article is generally much looser in rigor. Usually, only the main ideas of each argument are given. "Knot coloring" saves the reader quite a bit of mental checking and processing through its direct style. As a consequence, "knot coloring" should be readable to anyone with the minimum background required to begin studying knot theory.

Conclusion

"Knot coloring" is well-organized and direct. By building up the readers' knowledge of colorings in a step-by-step way, the authors establish a clear "big picture" understanding of the paper's logic. Carter, Silver and Williams recognize that their topic is well-suited to novice mathematicians, so the paper has rigorous, detailed arguments, placing less mental strain on their readers than those found in typical mathematical research articles. Further rhetorical analysis would likely reveal other ways "knot coloring" achieves clarity. For example, one could ask how other aspects of style, such as tense and stress positions, fluidly guide the reader's imagination from one idea to the next. "Knot coloring" is likely to act as a helpful introduction to research problems of knot theory to young mathematicians.

WORKS CITED

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